Eigenvalue Errors in the Method of Weighted Residuals

JACK G. CROUCH*
University of Dayton Research Institute, Dayton, Ohio
WILLIAM J. ANDERSON† AND DONALD T. GREENWOOD‡
The University of Michigan, Ann Arbor, Mich.

A theoretical study is made of the eigenvalue error which results from applying the method of weighted residuals to a certain type of linear, nonself-adjoint eigenvalue problem. The conclusions are valid for all self-adjoint problems and for nonself-adjoint problems with real eigenvalues as in certain panel flutter and follower load problems in subcritical load situations. The approximate eigenvalue is expressed as a Maclaurin series in terms of the error in the approximating functions relative to the exact eigenfunctions. The eigenvalue error is shown to be of second order when the error in the approximating functions is of first order. This result is true even if the weighting functions violate the adjoint boundary conditions to first order. To demonstrate, the problem of lateral oscillations of a missile under a follower thrust is solved in several different ways and the eigenvalue error is examined. The computations clearly support the theoretical results.

Nomenclature

$a_i(x)$ = coefficient in differential operator L_{1x}
a_{jr} = Fourier error coefficient in approximating function
A, A_∞ = matrices, $n \times n$ and $n \times \infty$
$B_s(\ ), B_s^*(\ )$ = operators in boundary conditions
$b_i(x)$ = coefficient in differential operator L_{2x}
b_{sv} = coefficient in $B_s(\ )$
b_{ir} = Fourier error coefficient in weighting function
B, B_∞ = matrices, $n \times n$ and $n \times \infty$
c_{sv} = coefficient in $B_s(\ )$
c_j = coefficient in assumed solution $\bar{u}$
C = matrix $[c_i]$, $n \times 1$
C_e = eigenvector matrix of weighted-residual matrix Eq. (28)
C_{eK} = column matrix Maclaurin coefficient of C_e
$c_{el\,e}$ = the eth element of C_{el}
$\det|a_{ij}|$ = determinant of matrix $[a_{ij}]$
$e_i(x), e_i^*(x)$ = error in approximating and weighting functions
EI = bending stiffness of uniform beam
$E_{ij}(\eta)$ = an integral in weighted-residual Eq. (44)
$F_{ij}(\eta)$ = an integral in weighted-residual Eq. (44)
h = assumed order of $L_{2x}(\ )$
I = identity matrix
K = nondimensional constant appearing in formula for ω_1
L_{1x}, L_{2x} = linear differential operators, see Eqs. (3) and (4)
L_{1x}^*, L_{2x}^* = linear differential operators adjoint to L_{1x}, L_{2x}, respectively
l = length of missile
m = assumed order of L_{1x}
M = linear manifold of functions, members of R, which satisfy boundary conditions (7)
n = order of weighted-residual approximation
P = missile engine thrust
R = linear vector space of functions $g(x)$ with square-integrable derivatives up to the order m inclusive and with the norm (8); $0 \leq \times \leq 1$
R' = linear vector space of piecewise-continuous, square-integrable functions $f(x)$ with the norm (11); $0 \leq \times \leq 1$
$\bar{t}$ = time, sec
t = nondimensional time; $t = \omega_1 \bar{t}$ where $\omega_1 = K\text{-}(EI/ml^4)^{1/2}$
$T(t)$ = time dependent part of separable solution
$u(x)$ = dependent variable in eigenvalue problem; missile lateral deflection, nondimensional
$\bar{u}(x)$ = assumed approximate solution of eigenvalue problem
$v(x)$ = dependent variable in the adjoint problem Eq. (40); also used to represent any function in R or R' Eq. (12).
$\bar{v}(x)$ = complex conjugate of $v(x)$, Eq. (12)
$w_i(x)$ = the ith weighting function, near the ith adjoint eigenfunction
$\bar{x}$ = distance along missile reference axis, ft
x = nondimensional distance ($x = \bar{x}/l$)
$\bar{z}(\bar{x},\bar{t})$ = lateral deflection of missile neutral axis, ft
$z(x,t)$ = nondimensionalized $\bar{z}$, ($z = \bar{z}/l$)
$\alpha(\bar{x},\bar{t})$ = distance a particular missile cross section, located at $\bar{x}$, has traveled parallel to the reference axis, in $\bar{t}$ sec
β^2 = Pl^2/EI = thrust parameter
δ_{ij} = the Dirac Delta function ($\delta_{ij} = 0$, $i \neq j$; $\delta_{ij} = 1$, $i = j$)
$\epsilon(x)$, $\epsilon(x,\eta)$ = equation residual or error in the equation resulting from substituting $\bar{u}$ for u
η = error magnitude
λ = eigenvalue
λ_i, λ_e = specific exact eigenvalues, order unspecified
$\bar{\lambda}_e$ = the eth approximate eigenvalue taken in some unspecified order
Λ, Λ_∞ = matrices $[\delta_{ij}\lambda_i]$, $n \times n$ and $\infty \times \infty$
λ_{eK} = the Kth Maclaurin coefficient in expansion of $\bar{\lambda}_e$
$\psi_i(x)$, $\psi_i^*(x)$ = the ith exact eigenfunction or ith exact adjoint eigenfunction
$\phi_j(x)$ = the jth approximating function, "near" the jth eigenfunction
$\theta_i(x)$ = the ith zero-thrust eigenfunction
ω_1 = the first bending frequency of the zero-thrust missile (beam)

Received January 5, 1970; revision received May 13, 1970.

* Research Engineer and Associate Professor of Mechanical Engineering. Member AIAA.
† Associate Professor of Aerospace Engineering. Member AIAA.
‡ Professor of Aerospace Engineering.

Introduction

GALERKIN'S method has been widely used to provide approximate solutions to boundary value problems because of its ease of application. Usually, the user assumes a solution in series form, involving a set of selected functions and an equal number of generalized coordinates or constants. When substituted into the governing equations, the series solution is not exact and some errors or "residuals" are obtained. The residual errors are required to be orthogonal to each of the set of selected functions in turn. This results

in a set of equations for the generalized coordinates, and a solution is obtained. As an alternative, orthogonality may be required with each member of a set of weighting functions different from the functions appearing in the assumed solution.

In other related techniques, the error may occur in the satisfaction of boundary conditions or in both the governing equation and the boundary conditions. Each method attempts to minimize this error. Crandall has given the name "method of weighted residuals" to these methods, which include the method of least squares, the collocation method, the subdomain method, the method of moments, the Galerkin method and many others.[1] Finlayson and Scriven have presented an excellent review of the method of weighted residuals including an extensive bibliography.[2] These latter authors stress the need for comparative studies among the methods and for determination of error bounds.

Our discussion will be restricted at first to the method as applied to a type of boundary value problem having two independent variables and one dependent variable. The differential equation is separable in the sense that it could be solved by the separation of variables technique as described by Friedman[3] and many others. The separated spatial differential system would then constitute a nonself-adjoint eigenvalue problem.[3]

Our initial objective is to apply the weighted-residual method to the separated eigenvalue problem for determination of the eigenvalue. This is equivalent to applying the method directly to the boundary value problem. Functions "near" the spatial eigenfunctions, and weighting functions "near" the eigenfunctions of the adjoint system are used. Certain continuity properties are required of all functions used. The weighting functions are not required, however, to satisfy the adjoint boundary conditions exactly. The results apply to the special case where the system is self-adjoint. The approach used will allow the resulting approximate eigenvalues to be expressed as Maclaurin expansions in terms of the error in the selected functions. The first three coefficients in the expansions will be examined. The eigenvalues, in this development, are assumed to be real.

Our second objective is to demonstrate the theoretical results by taking an actual problem—the lateral oscillations of a missile. The thrust of the missile is a follower force as described by Bolotin.[4] Three types of weighting functions will be used and the eigenvalue errors will be calculated for many different magnitudes of error in the selected functions.

The theory developed predicts that the errors in the eigenvalues will be of second order when the errors in the mode shapes are of first order. Roberts has come to essentially the same conclusion for a slightly simpler problem using a different approach.[5] He goes on to present an adjoint variational technique for determining the eigenvalues. However, Roberts requires his weighting functions to satisfy the adjoint boundary conditions.

Finlayson[6] has recently shown the equivalence of the weighted-residual method and an adjoint variational principle, again provided the weighting functions satisfy the adjoint boundary conditions. However, Finlayson's proof appears capable of being extended, to eliminate this restriction, and thus agree with the present result that the eigenvalue will be stationary even though the weighting functions do not satisfy the adjoint boundary conditions.

Rayleigh's principle describes the stationary property of eigenvalues for "conservative dynamical systems." Duncan shows the equivalence of the Galerkin method and Rayleigh's principle when both are applicable.[7] However, Duncan states that, in mechanical applications, the Galerkin method will yield second-order accuracy in the eigenvalues only when the weighting functions are chosen in a very specific way. According to Duncan, this choice renders the Galerkin method equivalent to the use of Lagrange's equations, and leads to the second-order accuracy.

The theory and calculations presented in this paper do not support Duncan's contention that special weighting functions are needed. In fact, the calculations show second-order eigenvalue accuracy for all three Galerkin approximations used. None of these approximations involved Duncan's special weighting functions and two of the approximations involved weighting functions which did not even satisfy the adjoint boundary conditions.

Mathematical Preliminaries

Consider the real, linear, eigenvalue problem:

$$L_{1x}(u) - \lambda L_{2x}(u) = 0 \quad 0 \leq x \leq 1 \tag{1}$$

$$B_s(u) = 0 \quad s = 1,2,\ldots m \tag{2}$$

where

$$L_{1x} = a_0(x)\frac{d^m}{dx^m} + a_1(x)\frac{d^{m-1}}{dx^{m-1}} + \ldots + a_{m-1}(x)\frac{d}{dx} + a_m(x) \tag{3}$$

$$L_{2x} = b_0(x)\frac{d^h}{dx^h} + b_1(x)\frac{d^{h-1}}{dx^{h-1}} + \ldots + b_{h-1}(x)\frac{d}{dx} + b_h(x) \tag{4}$$

$$a_0(x) \neq 0 \text{ for } 0 \leq x \leq 1 \tag{5}$$

$$m > h \tag{6}$$

and

$$B_s(u) = \sum_{\nu=0}^{m-1} \left\{ b_{s\nu}\frac{d^\nu u}{dx^\nu}\bigg|_{x=0} + c_{s\nu}\frac{d^\nu u}{dx^\nu}\bigg|_{x=1} \right\} = 0 \quad s = 1,2,\ldots m \tag{7}$$

Let us assume that the system 1) possesses a set of exact eigenfunctions $\{\psi_i(x);\ i = 1,2,\ldots\}$ and a set of corresponding exact eigenvalues $\{\lambda_i;\ i = 1,2,\ldots\}$. If this differential system is self-adjoint and L_{2x} is positive definite, then the eigenvalues are real and the eigenfunctions are orthogonal in a certain sense.[1] The system is not required to be self-adjoint, but the eigenvalues must be real and the eigenfunctions must form a basis for the manifold of functions satisfying Eq. (2). Our discussion shall be limited to real functions.

Consider the linear vector space R of all functions $g(x)$, $0 \leq x \leq 1$, with square integrable derivatives up through order m and with the norm

$$\|g(x)\| = \left[\int_0^1 (g)^2 dx\right]^{1/2} + \left[\int_0^1 (g')^2 dx\right]^{1/2} + \ldots + \left[\int_0^1 (g^{(m-1)})^2 dx\right]^{1/2} \tag{8}$$

Note that g can be obtained from g' by integration, g' from $g'', \ldots$ and g^{m-1} from g^m. Thus $g, g', \ldots g^{(m-1)}$ are all continuous in (0,1). One can speak, therefore, of the boundary values of $g, g', \ldots g^{(m-1)}$ at $x = 0$ and $x = 1$.

The set of all members of R which satisfy the boundary conditions, Eq. (2), forms a linear manifold M which includes the set $\{\psi_i(x);\ i = 1,2,\ldots\}$. Let us assume that the set $\{\psi_i(x);\ i = 1,2,\ldots\}$ spans M. Thus, any selected approximating function which belongs to M may be expressed as a generalized Fourier series in terms of the set $\{\psi_i(x);\ i = 1,2,\ldots\}$. This Fourier series converges in the norm (8).

If system (1) is nonself-adjoint, let the adjoint system be

$$L_{1x}^*(v) - \lambda L_{2x}^*(v) = 0 \qquad 0 \leq x \leq 1 \tag{9}$$

$$B_s^*(v) = 0 \qquad s = 1,2,\ldots m \tag{10}$$

where the operators are related to those in Eqs. (1) and (2) as described in Ref. 3, and where the adjoint boundary conditions B_s^* do not involve the eigenvalue λ. The adjoint eigenfunctions are denoted by $\{\psi_i^*(x);\ i = 1,2,\ldots\}$ and because the eigenvalues of the original problem are real, the eigenvalues of the adjoint problem are identical to those of the original problem $\{\lambda_i;\ i = 1,2,\ldots\}$.[3]

To provide a Fourier expression for the Galerkin weighting functions, let us define a new linear vector space R' of all piecewise-continuous, square-integrable functions $f(x)$, $0 \leq x \leq 1$, with the norm

$$||f(x)|| = \left[\int_0^1 (f)^2 dx\right]^{1/2} \tag{11}$$

Note that the functions $f(x)$ are not required to satisfy any particular boundary conditions. Thus, the set $\{\psi_i^*(x);\ i = 1,2,\ldots\}$ belongs to R'. Assume R' is spanned by the set $\{\psi_i^*(x);\ i = 1,2,\ldots\}$, in terms of which any such weighting function possesses a Fourier expansion converging in the norm (11).

It can be shown that the sets $\{\psi_i(x),\ i = 1,2,\ldots\}$ and $\{\psi_i^*(x),\ i = 1,2,\ldots\}$ may be normalized so that they satisfy a biorthogonality relation[8]

$$\begin{aligned}\langle\psi_i, L_{2x}^*(\psi_j^*)\rangle = \langle L_{2x}(\psi_i), \psi_j^*\rangle &= 0 \quad i \neq j \\ &= 1 \quad i = j\end{aligned} \tag{12}$$

where the inner product in R and R' is defined as:

$$\langle u,v\rangle = \int_0^1 uv\,dx \tag{13}$$

The Method of Weighted Residuals

To obtain approximate solutions for the eigenvalues $\{\lambda_i, i = 1,2,\ldots\}$, one forms an assumed solution

$$\bar{u}(x) = \sum_{j=1}^{n} c_j \phi_j(x) \tag{14}$$

where the linearly independent set $\{\phi_j(x);\ j = 1,2,\ldots n\}$ consists of selected functions which are members of M. The equation residual is the function

$$\epsilon(x) = L_{1x}(\bar{u}) - \mu L_{2x}(\bar{u}) \tag{15}$$

In the method of weighted residuals, one requires the residual to be orthogonal to each member of a linearly independent set of weighting functions $\{w_i(x);\ i = 1,2,\ldots n\}$ which belong to R'

$$\langle\epsilon, w_i\rangle = 0 \quad i = 1,2,\ldots n \tag{16}$$

Nontrivial solutions $\{c_j;\ j = 1,2,\ldots n\}$ will exist for this set of n equations if and only if the determinant of coefficients of the c_j vanishes. Values of λ for which the determinant vanishes are the approximate eigenvalues $\{\bar{\lambda}_i;\ i = 1,2,\ldots n\}$.

Let us now introduce the idea of "error" in the selected functions and weighting functions. If one uses the exact eigenfunctions and exact adjoint eigenfunctions in these roles, Galerkin's method leads to exact results (as discussed below). Consider

$$\phi_j(x) = \psi_j(x) + \eta e_j(x) \qquad j = 1,2,\ldots n \tag{17}$$

and

$$w_i(x) = \psi_i^*(x) + \eta e_i^*(x) \qquad i = 1,2,\ldots n \tag{18}$$

$$\eta \leq 0 \tag{19}$$

where $e_j(x)$ satisfies the original boundary conditions, $e_i^*(x)$ has arbitrary boundary values, and η represents the order of magnitude of the error. Then $\eta e_j(x)$ and $\eta e_i^*(x)$ are the "errors" in ϕ_j and w_i, respectively. If η vanishes, one obtains the exact solutions for the eigenvalues as follows. Equation (16) becomes

$$\left\langle L_{1x}\left(\sum_j^n c_j\psi_j\right) - \lambda L_{2x}\left(\sum_j^n c_j\psi_j\right), \psi_i^*\right\rangle = 0 \tag{20}$$
$$i = 1,2,\ldots n$$

and since

$$L_{1x}(\psi_j) = \lambda_j L_{2x}(\psi_j) \qquad j = 1,2,\ldots n \tag{21}$$

one can use Eq. (12) to obtain

$$\lambda_i - \lambda = 0 \qquad i = 1,2,\ldots n \tag{22}$$

Thus the approximate eigenvalues become exact when η vanishes.

Error Theorems

The hypotheses are now briefly summarized for the reader's convenience: a) A real, linear eigenvalue problem possessing a set of real eigenvalues and eigenfunctions is considered. b) The eigenfunctions form a basis for the manifold M of suitably differentiable functions which satisfy the boundary conditions. c) An adjoint system exists with boundary conditions not involving the eigenvalue. d) The adjoint eigenfunctions form a basis for the space, R', of piecewise-continuous, square-integrable weighting functions. e) Convergence in the norms (8) or (11) is implied as appropriate. f) The approximate eigenvalue and eigenvector are analytic functions of η, the order of magnitude of the error in the approximating functions.

Under the hypotheses stated, using the method of weighted residuals, the error in the approximate eigenvalues will be nearly proportional to η^2 as η becomes small. Proof: By virtue of hypotheses (b) and (d) and noting that the approximating functions are members of M and the weighting functions are members of R', one may write the following Fourier expansions:

$$\phi_j(x) = \psi_j(x) + \eta \sum_{r=1}^{\infty} a_{jr}\psi_r(x) \qquad j = 1,2,\ldots n \tag{23}$$

$$w_i(x) = \psi_i^*(x) + \eta \sum_{r=1}^{\infty} b_{ir}\psi_r^*(x) \qquad i = 1,2,\ldots n \tag{24}$$

One understands that series Eq. (23) converges in the norm Eq. (8) while series Eq. (24) converges in the norm Eq. (11). Thus, the equality sign in (24) does not imply that w_i satisfies the boundary conditions (10). By the method outlined in Ref. 8, it may be verified that the integrals appearing in Eq. (16) converge when expansions Eqs. (23) and (24) are used. The coefficients in Eqs. (23) and (24) may be determined by using the biorthogonality relation Eqs. (12).

One can now form Eq. (16) using Eqs. (14, 15, and 21), and expansions (23) and (24).

$$\langle L_{1x}(\bar{u}) - \lambda L_{2x}(\bar{u}), w_i\rangle = 0 \qquad i = 1,2,\ldots n$$

$$\left\langle L_{1x}\left(\sum_{j=1}^{n} c_j\phi_j\right) - \lambda L_{2x}\left(\sum_{j=1}^{n} c_j\phi_j\right), w_i\right\rangle = 0$$

$$\sum_{j=1}^{n} c_j \left\langle L_{2x}\left(\lambda_j\psi_j + \eta\sum_{r=1}^{\infty} a_{jr}\lambda_r\psi_r\right) - \lambda L_{2x} \times \left(\psi_j + \eta\sum_{r=1}^{n} a_{jr}\psi_r\right), \left(\psi_i^* + \eta\sum_{r=1}^{\infty} b_{ir}\psi_r^*\right)\right\rangle = 0$$

$$\sum_{j=1}^{n} c_j \left\{\left[\delta_{ij}\lambda_j + \eta(a_{ji}\lambda_i + b_{ij}\lambda_j) + \eta^2\sum_{r=1}^{\infty} a_{jr}\,b_{ir}\lambda_r\right] - \lambda\left[\delta_{ij} + \eta(a_{ji} + b_{ij}) + \eta^2\sum_{r=1}^{\infty} a_{jr}b_{ir}\right]\right\} = 0 \tag{25}$$
$$i = 1,2,\ldots n$$

$$\delta_{ij} = 0 \qquad i \neq j \tag{26}$$

$$\delta_{ij} = 1 \qquad i = j \tag{27}$$

Equation (25) may be written in matrix form as

$$[\Lambda + \eta(B\Lambda + \Lambda A^T) + \eta^2 B_\infty \Lambda_\infty A_\infty{}^T]C - \lambda[I + \eta(B + A^T) + \eta^2 B_\infty A_\infty{}^T]C = 0 \tag{28}$$

where the matrices appearing are listed with their orders: I = identity matrix, $n \times n$, $B = [b_{ij}]$, $n \times n$, $B_\infty = [b_{ij}]$, $n \times \infty$, $A = [a_{ij}]$, $n \times n$, $A_\infty = [a_{ij}]$, $n \times \infty$, $A^T = [a_{ji}]$, $n \times n$ (A transpose), $\Lambda = [\lambda_j \delta_{ij}]$, $n \times n$, $\Lambda_\infty = [\lambda_j \delta_{ij}]$, $\infty \times \infty$, $C = [c_j]$, $n \times 1$.

Let us suppose that $\bar{\lambda}_e$ and C_e are an eigenvalue of Eq. (28) and the corresponding eigenvector. Then, clearly, $\bar{\lambda}_e$ and C_e each depend upon η. When η is zero, $\bar{\lambda}_e$ reduces to λ_e and the transpose of C_e reduces to

$$C_e{}^T|_{\eta=0} = C_{e0}{}^T = K[0 \cdots 010 \cdots 0]$$

(where the unit element is in column e). K is arbitrary and it is taken to be unity. Since $\bar{\lambda}_e$ and C_e are analytic functions of η, they may be expressed as Maclaurin expansions

$$\bar{\lambda}_e = \lambda_e + \lambda_{e1}\eta + \lambda_{e2}\eta^2 + \ldots = \lambda_e + \sum_{K=1}^{\infty} \lambda_{eK}\eta^K \tag{29}$$

$$C_e = C_{e0} + C_{e1}\eta + C_{e2}\eta^2 + \ldots = \sum_{K=0}^{\infty} C_{eK}\eta^K \tag{30}$$

This perturbation method is described in Ref. 9.

Let us now substitute Eqs. (29) and (30) into Eq. (28)

$$[\Lambda + \eta(B\Lambda + \Lambda A^T) + \eta^2 B_\infty \Lambda_\infty A_\infty{}^T]\left[\sum_{K=0}^{\infty} C_{eK}\eta^K\right] - \left[\lambda_e + \sum_{K=1}^{\infty} \lambda_{eK}\eta^K\right]\left[I + \eta(B + A^T) + \eta^2 B_\infty A_\infty{}^T\right]\left[\sum_{K=0}^{\infty} C_{eK}\eta^K\right] = 0 \tag{31}$$

The coefficient of each power of η must vanish. Accordingly, the zero power of η is

$$\Lambda C_{e0} - \lambda_e I C_{e0} = 0, \quad \lambda_e = \lambda_e \tag{32}$$

For the first power of η,

$$\Lambda C_{e1} + (B\Lambda + \Lambda A^T)C_{e0} - \lambda_e I C_{e1} - \lambda_e(B + A^T)C_{e0} - \lambda_{e1} I C_{e0} = 0 \tag{33}$$

Let us examine row e of Eq. (33):

$$\lambda_e c_{e1e} + b_{ee}\lambda_e + \lambda_e a_{ee} - \lambda_e c_{e1e} - \lambda_e b_{ee} - \lambda_e a_{ee} - \lambda_{e1} = 0 \tag{34}$$

where c_{e1e} is element e of C_{e1}. Finally one obtains

$$\lambda_{e1} = 0 \tag{35}$$

and

$$\bar{\lambda}_e = \lambda_e + \lambda_{e2}\eta^2 + \lambda_{e2}\eta^3 + \ldots \tag{36}$$

This completes our proof. However, it is additionally shown in Ref. 8 that

$$\lambda_{e2} = \sum_{r=n+1}^{\infty} b_{er}a_{er}(\lambda_r - \lambda_e) \tag{37}$$

Lateral Deflection of a Missile

Let us now consider a specific problem in order to verify, numerically, the theoretical results just obtained. The problem chosen is the lateral deflection of a missile under the action of a "follower" thrust which remains tangent to the neutral axis at the point of application. For simplicity, the missile is treated here as a free-free, uniform beam. With the advent of large aerospace vehicles powered by very high-thrust engines, this problem has attracted attention. Because of the nonconservative force, oscillatory, elastic instability will occur at a sufficiently high thrust level. Of concern also is the probable effect of the lateral motions upon the gyroscopic sensors and, ultimately, upon the directional control of the vehicle.

Beal has calculated lateral oscillation frequencies using the Galerkin method for a wide range of thrust levels and has predicted those points at which coalescence of frequencies indicates instability.[10] He also introduced a simple control system in order to study parametric instability caused by longitudinal thrust pulsation. Here, only the basic missile without the control system and without the thrust pulsation will be considered. Our goal is to demonstrate how the error in an eigenvalue (related to the frequency) varies at a single thrust level as the errors in the assumed mode shapes increase.

Physical Description and Notation

The missile being considered is shown in Fig. 1. The notation used is similar to that used by Beal. Precise definition of the variables shown is given in the nomenclature.

Generally, the missile motions are referred to a hypothetical initial position in which no deformations have taken place. A given cross section is located a distance $\bar{x}$ from the missile tail. At a future time $\bar{t}$ the cross section has moved a distance α parallel to the reference line and a distance $\bar{z}$ laterally from the reference line. The reference line is the initial neutral axis. Note that P remains tangent to the neutral axis.

The missile is represented as a free-free, uniform beam, with stiffness EI and mass per unit length m. Simple Euler-Bernoulli beam theory is used. A separated solution $z(x,t) = u(x)\,T(t)$ is assumed, and the separation of variables technique yields the following differential system for the nondimensional spatial variable $u(x)$

$$u'''' + \beta^2[(1 - x)u']' - \lambda u = 0, \qquad u''' = u'' = 0,\ x = 0,1 \tag{38}$$

Beal[10] has calculated that the eigenvalues of Eq. (38) are real for β^2 less than 109.9. The temporal differential equation is

$$K^2\ddot{T} + \lambda T = 0 \tag{39}$$

Corresponding to the differential system of Eq. (38) is an adjoint differential system[8]:

$$v'''' + \beta^2[(1 - x)v']' - \lambda v = 0, \quad (v''' + \beta^2 v')|_{x=0} = 0, \quad (v'' + \beta^2 v)|_{x=0} = 0,\ v'' = v''' = 0 \text{ at } x = 1 \tag{40}$$

It is apparent that the differential system (38) is not self adjoint since its adjoint (40) differs from it. The mathematical property of nonself-adjointness is related to the

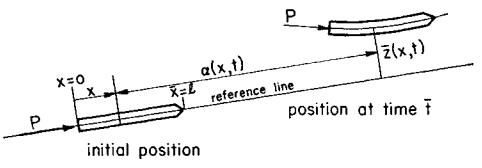

Fig. 1 Notation.

Table 1 Selected functions for weighted-residual approximations

Approximation	Approximating functions	Weighting functions
Type I	$\phi_1 = \psi_1 + \eta[-2\theta_1 - 3(1-x) - \psi_1]$ $\phi_2 = \psi_2 + \eta[0.167\theta_2 - 0.833(1-x) - \psi_2]$ $\phi_3 = \psi_3 + \eta[0.667\theta_3 - 0.333(1-x) - \psi_3]$ $\phi_4 = \psi_4 + \eta[0.7\theta_4 - 0.3(1-x) - \psi_4]$	$w_1 = \psi_1^* + \eta[\theta_1 + (0.405 + 3.01x)e^{-20x} - \psi_1^*]$ $w_2 = \psi_2^* + \eta[\theta_2 + (0.422 + 3.21x)e^{-20x} - \psi_2^*]$ $w_3 = \psi_3^* + \eta[\theta_3 + (0.436 + 3.37x)e^{-20x} - \theta_3^*]$ $w_4 = \psi_4^* + \eta[\theta_4 + (0.450 + 3.53x)e^{-20x} - \psi_4^*]$
Type II	(Same functions as above)	$w_1 = \psi_1^* + \eta[-2\theta_1 - 3(1-x) - \psi_1^*]$ $w_2 = \psi_2^* + \eta[0.167\theta_2 - 0.833(1-x) - \psi_2^*]$ $w_3 = \psi_3^* + \eta[0.667\theta_3 - 0.333(1-x) - \psi_3^*]$ $w_4 = \psi_4^* + \eta[0.7\theta_4 - 0.3(1-x) - \psi_4^*]$
Type III	(Same functions as above)	$w_1 = \psi_1^* + \eta[\theta_1 - \psi_1^*]$ $w_2 = \psi_2^* + \eta[\theta_2 - \psi_2^*]$ $w_3 = \psi_3^* + \eta[\theta_3 - \psi_3^*]$ $w_4 = \psi_4^* + \eta[\theta_4 - \psi_4^*]$

physical loss of energy conservation. The energy exchange represents work done by, or against, the thrust force caused by the interaction of the degrees of freedom. Such interaction first occurs when the thrust reaches a level corresponding to $\beta^2 = 109.9$, according to Beal. In the present paper $\beta^2 = 70$ is used to retain real eigenvalues.

The problem of the lateral deflections of a missile under the action of a "follower" thrust, as defined previously, satisfies all of the conditions needed to test the eigenvalue accuracy theorem. Let us proceed to find exact solutions for the eigenvalues, eigenfunctions and adjoint eigenfunctions. Following that, several approximating functions and weighting functions will be used in weighted-residual solutions for the eigenvalues. The errors in the eigenvalues can then be examined for verification of the theory.

"Exact" Solutions

The deflection eigenvalue problem (38) and its adjoint (40) were solved in double precision mode on the IBM 360-67 digital computer at the University of Michigan. The thrust parameter, β^2, was taken as 70 and the solutions were obtained using the fourth-order Runge-Kutta formulas in the modified form due to Gill as described by Ralston and Wilf.[11] These solutions are called "exact" in contrast to solutions obtained by the method of weighted residuals. The first four exact eigenvalues λ_i for the zero thrust case ($\beta^2 = 0$) are 0, 500.57, 3803.57, and 14,617.71. Also, the first four λ_i for the case $\beta^2 = 70$ are 0, 152.96, 2093.98, and 11,096.08. The zero thrust case is a self-adjoint problem with the eigenfunctions and adjoint eigenfunctions identical and denoted by $\{\theta_i(x);\ i = 1,2,3,4\}$. The eigenfunctions for $\beta^2 = 70$ somewhat resemble vibration modes of a beam. They will not be shown. In the next section, the weighted-residual method with three different kinds of weighting functions will be used to calculate approximate eigenvalues for comparison with the exact values.

Approximate Solutions

The three types of weighted-residual approximations used are shown in Table 1. Three term approximations were used in each case with approximating functions "near" ψ_2, ψ_3, and ψ_4 and with weighting functions "near" ψ_2^*, ψ_3^*, and ψ_4^*. The approximate eigenvalues $\bar{\lambda}_2$, $\bar{\lambda}_3$, and $\bar{\lambda}_4$ were calculated for many values of η for each type of approximation. The omission of $\bar{\lambda}_1$ is discussed below.

Note that the three types use the same approximating functions. These functions are written in the form of Eq. (17) being the sum of the exact eigenfunction and an error function multiplied by η. The approximating functions satisfy the boundary conditions of Eq. (38) and the error functions are linear combinations of the zero thrust eigentunctions, the exact eigenfunctions ψ_i and the function $(1 - x)$. The approximating functions with η equal to unity are shown in Fig. 2. These are roughly similar to the exact eigenfunctions.

The type I weighting functions satisfy the adjoint boundary conditions of Eq. (40). Again, these are written in the form of Eq. (18) as the sum of the exact adjoint eigenfunctions and an error function multiplied by η. The Type I weighting functions, with η equal to unity, are shown in Fig. 3.

The type II weighting functions do not satisfy the adjoint boundary conditions of Eq. (40). The error function used is the same as that in the approximating function except that ψ_i^* is substituted for ψ_i. Thus, when η is unity, the weighting functions and approximating functions are identical so that one has a Galerkin approximation in the usual sense.

The type III weighting functions again do not satisfy the adjoint boundary conditions. When η is unity, the weighting functions become simply the zero-thrust eigenfunctions.

Computation of Eigenvalues

Considering the first eigenvalue (zero) as already known exactly, let us examine approximations to λ_2, λ_3, and λ_4. Note that the theory does not require that the eigenvalues and eigenfunctions be ordered or that none be omitted. Consequently, one may use a three-term approximation in which the approximate mode shapes are near ψ_2, ψ_3, and ψ_4. This will lead to corresponding approximate eigenvalues of the differential system described in Eq. (38), defining the lateral motions of the missile. The three weighted-residual approximations just described will be used. Let us assume an approximate solution for the lateral deflection of the

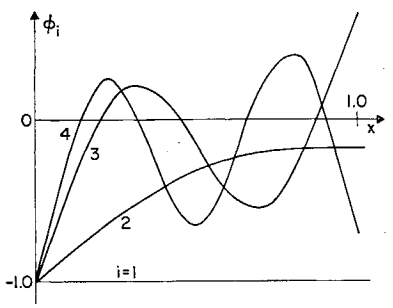

Fig. 2 Approximating functions, $\eta = 1$.

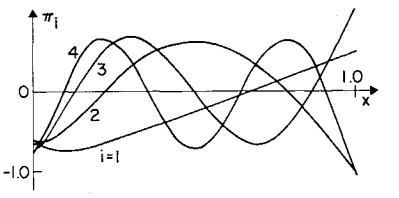

Fig. 3 Type I weighting functions, $\eta = 1$.

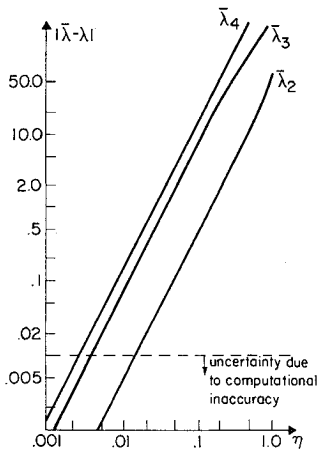

Fig. 4 Eigenvalue error, type I.

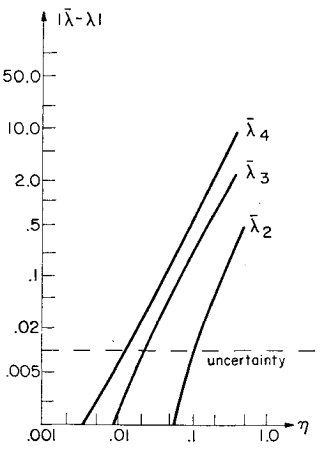

Fig. 6 Eigenvalue error, type III.

form

$$\bar{u}(x,\eta) = \sum_{j=2}^{4} c_j \phi_j(\eta,x) \tag{41}$$

where the functions $\phi_j(\eta,x)$ are those described previously and the c_j are unknown constants which can be determined from subsequent equations.

If u in Eq. (38) is replaced by $\bar{u}$,

$$\sum_{j=2}^{4} c_j(\{\phi_j'''' + \beta^2[(1-x)\phi_j']'\} - \bar{\lambda}\phi_j) = \epsilon(x,\eta) \tag{42}$$

The weighted residual method prescribes that ϵ be made orthogonal to a set of weighting functions. Using the weighting functions described previously,

$$\int_0^1 \epsilon(x,\eta) w_i(x,\eta) dx = 0 \quad i = 2,3,4 \tag{43}$$

which leads to

$$\sum_{j=2}^{4} c_j[E_{ij}(\eta) - \bar{\lambda}F_{ij}(\eta)] = 0 \quad i = 2,3,4 \tag{44}$$

where these definitions hold:

$$E_{ij}(\eta) = \int_0^1 \{\phi_j'''' + \beta^2[(1-x)\phi_j']'\} w_i dx, \text{ and}$$

$$F_{ij}(\eta) = \int_0^1 \phi_j w_i dx$$

When the condition for nontrivial solutions is applied, one has an equation defining the eigenvalues as functions of η

$$\det|E_{ij}(\eta) - \bar{\lambda}F_{ij}(\eta)| = 0 \tag{45}$$

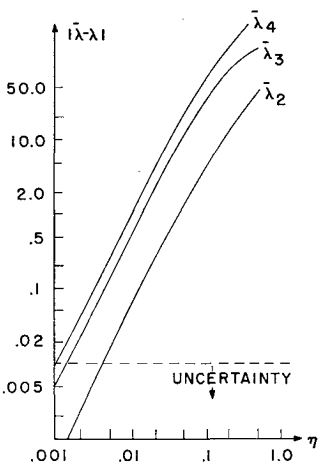

Fig. 5. Eigenvalue error, type II.

If one specifies the value of η, the corresponding values of $\bar{\lambda}$ may be computed from Eq. (45). Digital computation was employed to yield results for $0.0001 \leq \eta < 0.0020$ at increments of 0.0001 and for $0.002 \leq \eta < 0.368$ at increments of 0.002.

The results of the eigenvalue calculations are shown in Figs. 4, 5, and 6. The eigenvalue error magnitude $\bar{\lambda} - \lambda$ is plotted as a function of η with logarithmic scales used for both quantities. Our theory states that

$$\bar{\lambda}_e = \lambda_e + \lambda_{e2}\eta^2 + \lambda_{e3}\eta^3 + \ldots; \quad e = 2,3,4 \tag{46}$$

when the weighted-residual method is used. When η is small, one should have approximately

$$\log|\bar{\lambda}_e - \lambda_e| = \log|\lambda_{e2}| + 2\log\eta \tag{47}$$

The curves for all three approximations have slopes nearly equal to two, thus agreeing with Eq. (47) and with the theorem. The curves are shown only for the larger values of eigenvalue error. The results are inconclusive for errors less than 0.01 because the uncertainty in the exact eigenvalue has this same order of magnitude. Note that the smallest errors occur in the type III case.

Additional Computation

Further cases have been studied in which the error in the weighting function was eliminated but that in the approximating function retained.[8] The results are shown in Fig. 7. The reverse case was also computed, showing no eigenvalue error. These results are compatible with the theory but were not expected since the higher Maclaurin coefficients λ_{e3}, λ_{e4}, etc., have not yet been examined.

Finally, there is some question about the presence of a short wavelength "ripple" in the weighting functions. If such a ripple were erroneously introduced (as in an analog computer application) one might wonder if the error in the eigenvalue would be large. Three cases were studied in

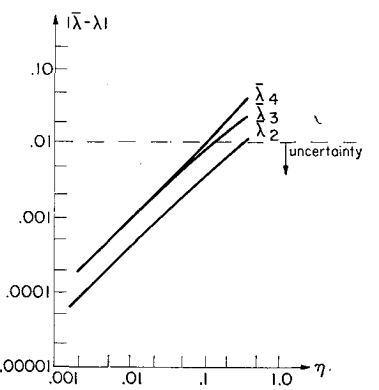

Fig. 7 Effect of zero error in weighting functions.

which a short wavelength ripple was added to the weighting function and the error analyzed. The resulting eigenvalue error was relatively small and was predicted by the theory.[8]

Conclusions

It has been shown theoretically and numerically that the eigenvalue error is of second order when the weighted-residual method is applied to the limited class of problems described. This second-order accuracy is obtained even though the weighting functions violate the adjoint boundary conditions to first order. The best accuracy in the example involved weighting functions which did not satisfy the adjoint boundary conditions.

Clearly the "nearness" of the assumed functions to the exact eigenfunctions and adjoint eigenfunctions is critical. In the example, the eigenvalue error was essentially zero if either the approximating function error or weighting function error was eliminated.

Based on these results, one should be able to proceed with confidence to calculate approximate eigenvalues and eigenfunctions by this method. One such situation would involve a parameter study in which eigenvalues and eigenfunctions for one solution could be used as approximations for the succeeding solution. Or, it may be possible to develop iterative methods where convergence could be established once one is in the proper neighborhood of the exact solution.

More work needs to be done toward rationalizing the selection of approximating functions and weighting functions, particularly in nonlinear applications. The results in the present paper need to be extended to include complex eigenvalues. Also, the higher order Maclaurin coefficients λ_{e3}, λ_{e4}, etc., should be determined.

The method of weighted residuals is a useful tool for finding approximate solutions for many kinds of differential equations. The Galerkin method, in particular, is easy to apply and need have no connection with any variational principle. It is suitable for both analog and digital computation and it may be used when the solution must be presented in analytic form. However, much remains to be learned and many areas invite research beyond the few specific topics mentioned here.

References

[1] Crandall, S. H., *Engineering Analysis*, McGraw-Hill, New York, 1956.

[2] Finlayson, B. A. and Scriven, L. E., "The Method of Weighted Residuals—A Review," *Applied Mechanics Reviews*, Vol. 19, No. 9, Sept. 1966, pp. 735–748.

[3] Friedman, B., *Principles and Techniques of Applied Mathematics*, Wiley, New York, 1956.

[4] Bolotin, V. V., *Non-Conservative Problems of the Theory of Elastic Stability*, Macmillan, New York, 1963, pp. 1–24.

[5] Roberts, P. H., "Characteristic Value Problems Posed by Differential Equations Arising in Hydrodynamics and Hydromagnetics," *Journal of Mathematical Analysis and Application*, Vol. 1, 1960, pp. 195–214.

[6] Finlayson, B. A., "Convective Instability of Ferromagnetic Fluids," *Journal of Fluid Mechanics*, Vol. 40, March 1970, pp. 753–758.

[7] Duncan, W. J., "The Principles of the Galerkin Method," R&M 1848, 1938, Technical Report of the Aeronautical Research Committee, Vol. II, Great Britain.

[8] Crouch, J. G., "Eigenvalue Errors in the Galerkin Method," Ph.D. dissertation, 1969, University Microfilms, The University of Michigan, Ann Arbor, Mich.

[9] Courant, R. and Hilbert D., *Methods of Mathematical Physics*, Vol. 1, Interscience, New York, 1953.

[10] Beal, T. R., "Dynamic Stability of a Flexible Missile Under Constant and Pulsating Thrusts," *AIAA Journal*, Vol. 3, No. 1, March 1965, pp. 486–494.

[11] Ralston, A. and Wilf, H. S., *Mathematical Methods for Digital Computers*, Wiley, New York, 1960, pp. 110–120.

[12] Kaplan, S., "On the Best Method for Choosing the Weighting Functions in the Method of Weighted Residuals," *Transactions of the American Nuclear Society*, Vol. 6, 1963.